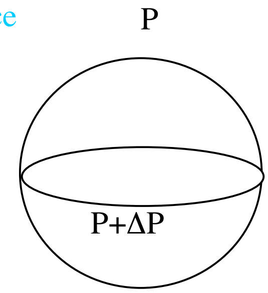
26. (12 points) Red-cell lysis pressure.
At issue in this problem is the degree of osmotic imbalance allowed by the red-cell membrane. If the red cell is exposed to a hypotonic environment (one with a lower concentration of osmolites than occurs inside the cell), water will flow into the cell until it swells to spherical and, if the osmotic imbalance is large enough, lyses (ruptures) and then reseals due to the line tension. What is the largest osmotic imbalance in moles/litre which the red cell can withstand without rupture?
Data:
Stretch modulus $K_s = 0.16\ J/m^2$; lysis occurs at 2% area expansion.
Red-cell membrane area=140 μm^2
Hint: See Tutorial 8.1 and Lecture 20.1.

Once the red cell has swollen to spherical, osmotic pressure can build up. The relation between the pressure difference and the surface tension is derived in Tutorial 8, which I copy here.
Question: how much membrane tension does a pressure difference ΔP produce in a spherical membrane of radius R?



The net force on the upper hemisphere must be zero.
The upward force due to the pressure difference is $\pi R^2 \Delta P$.
The downward force due to the membrane tension action around the equator is $2\pi R\tau$, where τ is the membrane tension.
Equating gives: $\pi R^2 \Delta P = 2\pi R \tau \quad \Rightarrow \quad \boxed{\Delta P = \frac{2\tau}{R}}$.

In order for the cell not to lyse, $\Delta P_{osmotic} = \Delta c_s k_B T < \frac{2\tau_{max}}{R}$, where τ_{max} is the maximum tension the cell membrane can have before it lyses.

Thus, $(\Delta c_s)_{max} = \frac{2\tau_{lysis}}{k_B T R}$.

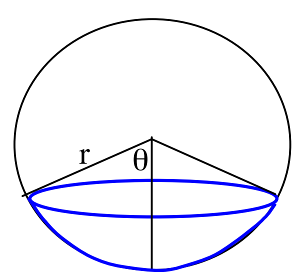
The lysis tension follows from $\tau = K_s\left(\frac{\Delta A}{A_0}\right) = 0.16(0.02) = 3.2\times 10^{-3}\ N/m$.

The radius of the cell follows from the area (you can put in the 2% expansion, if you want, but it is a small effect): $R = \left(\frac{A}{4\pi}\right)^{1/2} = \left(\frac{140\times 10^{-12}(1.02)}{4\pi}\right)^{1/2} = 3.37\times 10^{-6}\ m$.

Thus, $(\Delta c_s)_{max} = \frac{2\tau_{lysis}}{k_B T R} = \frac{(2)3.2\times 10^{-3}}{1.38\times 10^{-23}(300)3.37\times 10^{-6}} = 4.59\times 10^{23}$ particles/m^3, which converts to $\boxed{7.6\times 10^{-4}\ M}$.

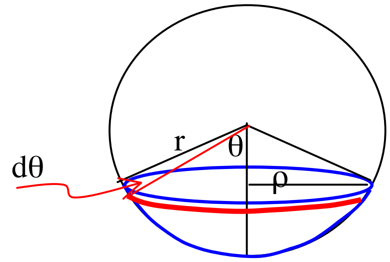
27.. (12 points) Membrane closure due to edge tension
Consider a flat circular patch of membrane of radius R. Assume that the membrane material is symmetrical, so the membrane is flat in its relaxed state and has a bending energy given by

$E_b = \frac{\kappa_b}{2}\int_S dA\left(\frac{1}{R_1} + \frac{1}{R_1}\right)^2$. This patch is surrounded by an open edge which has an energy $E_{edge} = 2\pi R\lambda$, where λ is the edge energy/unit length. In this problem, you will explore the conditions under which the edge tension is sufficient to force the patch to close up into a sphere.



To make the problem tractable, assume that the patch always has the shape of a spherical cap, as shown at the right, so that the open edge remains circular with radius $\rho = r\sin\theta$.

(a) Show that the area of the cap is related to r and θ by $A = \pi R^2 = 2\pi r^2(1-\cos\theta)$. Since A is fixed, this is a relation between r and θ.



(a) (4 points) The area element between the blue and red circles is $dA = 2\pi\rho r d\theta = 2\pi r^2 \sin\theta d\theta$.

Thus, the cap area from θ=0 to θ is

$$A \equiv \pi R^2 = 2\pi r^2 \int_0^\theta d\bar{\theta}\sin\bar{\theta} = 2\pi r^2(1-\cos\theta).$$

Note that for θ=π this becomes $4\pi r^2$, i.e., it gives correctly the area of the sphere.

For part (b), it is convenient to take this result one step further. Thus,

$$R^2 = 2r^2(1-\cos\theta) = 4r^2\sin^2\left(\frac{\theta}{2}\right), \text{so } \boxed{R = 2r\sin\left(\frac{\theta}{2}\right).}$$

(b) Show that the energy of the patch (including both the edge energy and the bending energy) can be written as $E = 8\pi\kappa_b \sin^2\frac{\theta}{2} + 2\pi R\lambda\cos\frac{\theta}{2}$.

(b) (4 points) Since the radii of curvature of the spherical surface are both r, we have

$$E = \frac{\kappa_b}{2}\int_S dA\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2 + \lambda(\text{perimeter}) = \frac{\kappa_b}{2}A\left(\frac{2}{r}\right)^2 + \lambda 2\pi\rho = 4\pi\kappa_b(1-\cos\theta) + 2\pi\lambda r\sin\theta$$

$$= 8\pi\kappa_b\sin^2\left(\frac{\theta}{2}\right) + 4\pi\lambda r\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \boxed{8\pi\kappa_b\sin^2\left(\frac{\theta}{2}\right) + 2\pi\lambda R\cos\left(\frac{\theta}{2}\right)},$$

where in the last line I have used the identities: $1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$ in the first term and $\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ in the second (plus the boxed identity from (a)).

(c) Show that there is a critical patch size $R_c = \frac{8\kappa_b}{\lambda}$ such that the patch will close to a sphere for $R>R_c$. What happens for $R<R_c$? Estimate the value of R_c. ($\kappa_b \sim 25\ k_BT; \lambda = 30\ k_BT/nm$)

(c) (4 points) The point here is to understand $E(\theta)$ for the interval $0 \le \theta \le \pi$. The system will, of course, find its lowest energy, so we need to find where that minimum is. As a start, notice:

- $\begin{matrix} E(0) = 2\pi R\lambda \\ E(\pi) = 8\pi\kappa_b \end{matrix}$, so $E(0) < E(\pi)$ for $R < \frac{1}{2}R_c$ and the opposite for for $R > \frac{1}{2}R_c$.
- Near θ=0, expand: $E(\theta) = 2\pi R\lambda + \pi\lambda(R_c - R)\left(\frac{\theta}{2}\right)^2 + \ldots$, so θ=0 is a local minimum for $R < R_c$.
- Calculate the derivative: $\frac{dE}{d\theta} = \pi\lambda\sin\left(\frac{\theta}{2}\right)\left[R_c\cos\left(\frac{\theta}{2}\right) - R\right]$, from which it follows that $\frac{dE}{d\theta} = 0$ at $\theta = 0$ and at $\cos\left(\frac{\theta}{2}\right) = \frac{R}{R_c}$, which can only be satisfied at $R \le R_c$.

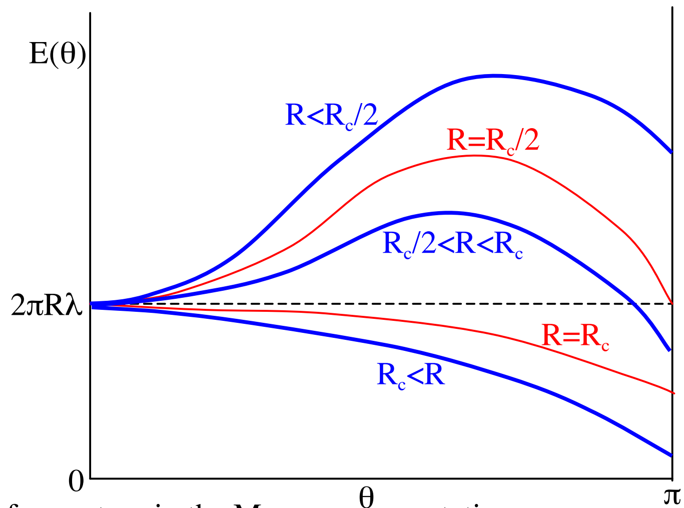
- Finally, note that $\left.\frac{dE}{d\theta}\right|_{\theta=\pi} = -\pi R\lambda$, so $\theta = \pi$ (i.e., the closed sphere) is always locally stable.

From these facts, we can conclude that:

- For $R < \frac{R_c}{2}$ the flat patch is the (stable) lowest-energy state but the sphere is also a local energy minimum (although with a higher energy).
- For $\frac{R_c}{2} < R < R_c$ the flat patch remains locally stable ("metastable") but it is not the ground state: the closed sphere has a lower energy.
- Finally, for $R_c < R$ the flat patch is unstable and the only stable configuration is the closed sphere.

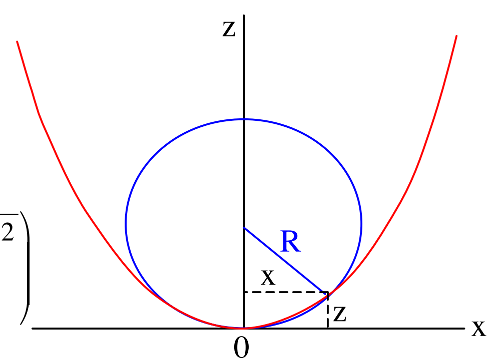
The graph at the right shows schematically the shape of the energy curve.

Note:
$$R_c = \frac{8\kappa_b}{\lambda} = \frac{8(25)}{30} = 6.67\ nm$$



28. (10 points) Principal radii of curvature in the Monge representation.
(a) I told you in class (Lect. 32.2) that, when the height function z(x,y) has the form $z = \frac{1}{2R_x}x^2 + \frac{1}{2R_y}y^2$, then R_x and R_y are the radii of curvature of the surface in the xz and yz planes. Prove that this is true. These are by definition the principle radii of curvature $R_{1,2}$ at the point x=y=0.

(a) (5 points) The question is "what radius should the blue circle have so that is best approximates the red curve in the vicinity of the origin?"
The equation of the circle in x, z coordinates is
$R^2 = x^2 + (R-z)^2$.
Solve this for z:



$$(R-z)^2 = R^2 - x^2 \quad \Rightarrow \quad z = R - \sqrt{R^2 - x^2} = R\left(1 - \sqrt{1 - \left(\frac{x}{R}\right)^2}\right)$$

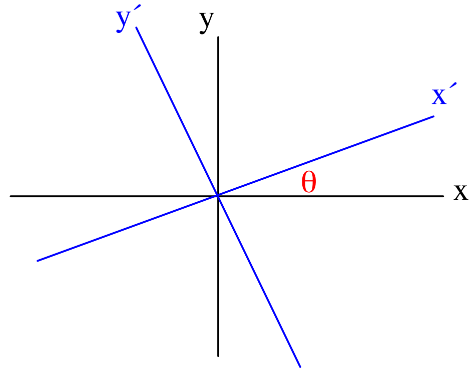
Now expand for small x:

$$z = R\left(1 - \sqrt{1 - \left(\frac{x}{R}\right)^2}\right) = R\left[1 - \left(1 - \frac{1}{2}\left(\frac{x}{R}\right)^2 + O\left(x^4\right)\right)\right] = \boxed{\frac{1}{2R}x^2} + O\left(\frac{x^4}{R^3}\right).$$

(b) Consider a surface described in the Monge representation by $z(x,y) = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2$.
What are the principal radii of curvature of this surface at the point x=y=0.

(b) (5 points) What I was expecting you to do here was to rotate coordinates to get to a frame in which the quadratic form is diagonal and then, in that new frame, to use the result of (a). Thus a general rotation of coordinates takes the form,

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta,$$



so, substituting,

$$z(x',y') = \frac{a'}{2}x'^2 + b'x'y' + \frac{c'}{2}y'^2 \text{ with}$$
$$a' = a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta$$
$$b' = -a\cos\theta\sin\theta + b\left(\cos^2\theta - \sin^2\theta\right) + c\cos\theta\sin\theta$$
$$c' = -a\sin^2\theta - 2b\cos\theta\sin\theta + c\cos^2\theta.$$

We now want to choose θ so as to set b´=0:

$$b\left(\cos^2\theta - \sin^2\theta\right) = \boxed{b\left(1 - 2\sin^2\theta\right)} = (a-c)\sin\theta\cos\theta = \boxed{(a-c)\sin\theta\sqrt{1-\sin^2\theta}}.$$

Squaring both sides now allows you to solve a quadratic equation for θ,

$$\sin^2\theta = \frac{1}{2}\left[1 \pm \frac{(a-c)}{\sqrt{4b^2 + (a-c)^2}}\right].$$ the two signs correspond to the possible rotation angles

When this result is substituted back into the formulas for a´ and c´, the result is,

$$a',c' = \frac{1}{2}\left[(a+c) \pm \sqrt{4b^2 + (a-c)^2}\right].$$ one sign choice gives a´ and the other gives c´.

It now follows from (a) that the two "principle radii of curvature" are:

$$\boxed{R_1,R_2 = \frac{1}{a},\frac{1}{c} = \frac{2}{(a+c) \pm \sqrt{4b^2 + (a-c)^2}}}.$$

I have omitted some (tedious) algebra here.

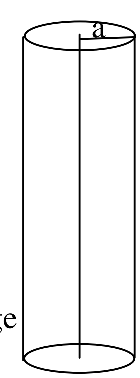
Comment:

If you know some differential geometry (and some of you seem to), you can do this by simply calculating the eigenvalues of the curvature tensor:

$$C = \begin{pmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$ with eigenvalues given by $(\lambda - a)(\lambda - c) - b^2 = 0$.

Solving for λ gives $\lambda = \frac{1}{2}\left[(a+c) \pm \sqrt{4b^2 + (a-c)^2}\right]$ as the principle curvatures, the reciprocals of which are the principal radii of curvature. (But, this is not what I expected you to do.)

29. (16 points) Long charged cylinder with counterions (Manning condensation)



In this problem, you will solve (exactly!) the problem of a line charge (like DNA) in aqueous solution.

Consider an infinitely long thin cylinder of radius a with a (negative) symmetrically distributed surface charge –λ per unit length immersed in water solution. A density $n(r)$ of corresponding positive neutralizing charges q has been released into solution.

The problem is to use the Poisson-Boltzmann equation to calculate the equilibrium charge distribution,

$\rho(r) = qn(r) = qn_0 e^{-\frac{q\Phi}{k_B T}}$. (see Lect. 29.1—29.3 for 1D analog).

(a) (3 points) Use Gauss's theorem to calculate the electric field $E(r)$ for $r \geq a$ in terms of λ and ρ(r).

(a) Gauss's law: $AE = 2\pi rLE = \frac{L}{\varepsilon}\left(-\lambda + \int_a^r d\bar{r}\, 2\pi\bar{r}\rho(\bar{r})\right)$, so $E(r) = \frac{1}{2\pi\varepsilon r}\left(-\lambda + \int_a^r d\bar{r}\, 2\pi\bar{r}\rho(\bar{r})\right)$.

Comment: For future reference, note the $r \to \infty$ limit here,

$rE(r) \underset{r\to\infty}{\to} \frac{1}{2\pi\varepsilon}\left(-\lambda + \int_a^\infty d\bar{r}\, 2\pi\bar{r}\rho(\bar{r})\right) = \frac{1}{2\pi\varepsilon}\left(-\lambda + Q_{screening}\right)$, where $Q_{screening}$ is the total charge per unit length of the counterion screening cloud. If screening is complete, then the right hand side vanishes. If there is not screening, then $Q_{screening}=0$. Partial screening leads to intermediate values.

(b) From (a), show that the electric field at r=a is $E(a) = \frac{-\lambda}{2\pi\varepsilon a}$.
This will be come a boundary condition in what follows.

(b) (2 points) For r=a, the density term has no contribution, so $E(a) = \frac{-\lambda}{2\pi\varepsilon a}$.

(c) Again, starting from (a) and using the fact that $E(r) = -\frac{d\Phi}{dr}$, show that the electric potential satisfies $\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = -\frac{\rho(r)}{\varepsilon}$. This is just the Poisson equation with cylindrical symmetry. With the substitution of the expression for ρ(r) given above, it becomes the Poisson-Boltzmann equation.

(c) (3 points) Recall that $\vec{E} = -\vec{\nabla}\Phi$, so in cylindrical geometry $E(r) = -\frac{d\Phi}{dr}$.

Thus, from (a), $-E(r) = \frac{d\Phi}{dr} = \frac{1}{2\pi\varepsilon r}\left(\lambda - \int_a^r d\bar{r}\, 2\pi\bar{r}\rho(\bar{r})\right)$, so $r\frac{d\Phi}{dr} = \frac{1}{2\pi\varepsilon}\left(\lambda - \int_a^r d\bar{r}\, 2\pi\bar{r}\rho(\bar{r})\right)$.

Taking the radial derivative gives $\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = \frac{1}{2\pi\varepsilon}\cdot\left(-2\pi r\rho(r)\right) \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = -\frac{\rho(r)}{\varepsilon}$.

(d) Show that, by making the substitutions $u = \ln\frac{r}{a}$ and $\phi(r) = \frac{q\Phi}{k_B T} - 2u$, the Poisson-Boltzmann equation can be written $\frac{d^2\phi}{du^2} = -\frac{n_0 q^2 a^2}{\varepsilon k_B T}e^{-\phi}$. This equation now has the 1D form.

(d) (2 points) Notice that $u \equiv \ln\frac{r}{a} \Rightarrow du = \frac{1}{r}dr$, so

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = \frac{1}{r^2}r\frac{d}{dr}\left(\frac{d\Phi}{du}\right) = \frac{1}{a^2}\left(\frac{a}{r}\right)^2\frac{d^2\Phi}{du^2} = \frac{e^{-2u}}{a^2}\frac{d^2\Phi}{du^2}.$$

Furthermore, $\rho(r) = qn(r) = qn_0 e^{-\frac{q\Phi}{k_B T}}$, so finally

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = \frac{1}{a^2}\left(\frac{a}{r}\right)^2\frac{d^2\Phi}{du^2} = \frac{e^{-2u}}{a^2}\frac{d^2\Phi}{du^2} = -\frac{qn_0}{\varepsilon}e^{-\frac{q\Phi}{k_B T}} \Rightarrow \frac{d^2\Phi}{du^2} = -\frac{qa^2 n_0}{\varepsilon}e^{-\frac{q\Phi}{k_B T}+2u}.$$

Comment: The choice of the zero of Φ is arbitrary. This arbitrariness is equivalent to a multiplicative factor in the definition of the normalization constant n_0. I will chose $\Phi = 0$ at r=a, which removes all arbitrariness.

Now, multiply through by $\frac{q}{k_B T}$: $$\frac{d^2\left(\frac{q\Phi}{k_B T} - 2u\right)}{du^2} = \boxed{\frac{d^2\phi}{du^2}} = -\frac{q^2 a^2 n_0}{\varepsilon k_B T} e^{-\frac{q\Phi}{k_B T}+2u} = \boxed{-\frac{q^2 a^2 n_0}{\varepsilon k_B T} e^{-\phi}}$$

(e) Integrate this equation (as in Lect. 29) to find the solution,

$\frac{q\Phi}{k_B T} = 2\ln\frac{r}{a} + 2\ln\left(1 + \sqrt{\frac{n_0 q^2 a^2}{2\varepsilon k_B T}}\ln\frac{r}{a}\right)$, where I have used the boundary condition that $\Phi(a) = 0$.

Be careful about signs.

(e) (2 points) Multiply both sides by $\frac{d\phi}{du}$:

$$\frac{d}{du}\left(\frac{1}{2}\left(\frac{d\phi}{du}\right)^2\right) = \frac{d^2\phi}{du^2}\cdot\frac{d\phi}{du} = -\frac{q^2 a^2 n_0}{\varepsilon k_B T} e^{-\phi}\cdot\frac{d\phi}{du} = \frac{d}{du}\left(\frac{q^2 a^2 n_0}{\varepsilon k_B T} e^{-\phi}\right).$$

Thus, $\frac{1}{2}\left(\frac{d\phi}{du}\right)^2 = \frac{q^2 a^2 n_0}{\varepsilon k_B T} e^{-\phi} + C$, where C is a constant of integration.

Comment:

I expected you to proceed here, as I did in class, under the ***hypothesis*** that, as $r \to \infty \quad (u \to \infty)$, $\phi(u)$ diverges and $\frac{d\phi}{du}$ goes to zero, and I will follow that route in a moment. However, I want you to stop here and notice that (unlike 1D) this assumption is far from obvious: It is certainly true that $\Phi(a) = 0$ by our convention and that $\frac{d\Phi}{dr} > 0$, as can be seen from (c), since the total number of counterions in the screening cloud $\rho(r)$ cannot be more than λ per unit length. From this it follows that, although $\Phi(r)$ may go to a constant as $r \to \infty$, it certainly cannot be decreasing. BUT, $\phi = \frac{q\Phi}{k_B T} - 2u$, so it is far from clear that ϕ is diverging as u gets large. In order for that to be true, $\frac{q\Phi}{k_B T}$ must be increasing *faster* than $2\ln\frac{r}{a}$. Indeed,

$\frac{d\phi}{du} = \frac{q}{k_B T}\left(r\frac{d\Phi}{dr}\right) - 2 = -\left(\frac{q}{k_B T} rE(r) + 2\right)$. Now, we know how $rE(r)$ behaves at large r (see comment at (a)), so $\left.\frac{d\phi}{du}\right|_{u\to\infty} = -\left(\frac{q}{k_B T}\left(rE(r)\right)_{r\to\infty} + 2\right) = \frac{q}{2\pi\varepsilon k_B T}\left[\lambda - Q_{screening} - \lambda_c\right]$,

$\lambda_c \equiv \frac{4\pi\varepsilon k_B T}{q} = \frac{q}{\ell_B}$ is the critical charge density. Thus, the vanishing of $\frac{d\phi}{du}$ at $u \to \infty$ is closely tied to the value of the $Q_{screening}$.

Anyway, we proceed now under the assumption that $\phi(u)$ diverges and $\frac{d\phi}{du}$ goes to zero, as before. We will find when we are finished that the assumption is consistent only for $\lambda > \lambda_c$.

Under this assumption, C=0, and we can solve to get $\frac{1}{2}e^{\phi/2}\frac{d\phi}{du} = \sqrt{\frac{n_0 q^2 a^2}{2\varepsilon k_B T}}$, which integrates to

$$e^{\phi/2} = A + u\sqrt{\frac{n_0 q^2 a^2}{2\varepsilon k_B T}}$$, where A is another constant of integration.

But, $\phi(u=0) = \Phi(r=a) = 0$, so A=1. Taking logarithms gives the result quoted in the problem. Notice at this point that the normalization n_0 remains to be determined from the boundary condition on E(a) from part (b).

Provided that boundary condition can be satisfied, it is evident that the solution we have found is consistent with the hypotheses that $\phi(u)$ diverges and $\frac{d\phi}{du}$ goes to zero.

(f) It remains to apply the boundary condition from (b) to determine the normalization constant n_0 for the density distribution. Do this—carefully! Show that there is a consistent solution with $n_0 > 0$ if and only if $\lambda > q/\ell_B$, where $\ell_B \equiv \frac{q^2}{4\pi\varepsilon k_B T}$ is called the Bjerrum length (PKT p. 340). Find this solution.

Comment: The only solution for $\lambda \le q/\ell_B$ is n_0=0. The meaning of this is that, for charge densities λ below this critical density, none of the counterions remain bound near the line charge. Instead, they have all wandered off to infinite distances. This is an entropic effect due to the infinite volume; it is analoguous to what happens to an atomic or molecular bound state in infinite volume (it always ionizes at any T>0).

(f) (2 points) Starting from the result (e) we calculate:

$$\frac{q}{k_B T}\frac{d\Phi}{dr}\bigg|_a = \frac{q}{k_B T}\left(\frac{\lambda}{2\pi\varepsilon a}\right) = \frac{2}{a} + \frac{2}{a}\cdot\sqrt{\frac{n_0 q^2 a^2}{2\varepsilon k_B T}}.$$

This is the equation that must be solved for n_0:

(note that the square root is an intrinsically positive quantity.)

$$\sqrt{\frac{n_0 q^2 a^2}{2\varepsilon k_B T}} = \left(\frac{q\lambda}{4\pi\varepsilon k_B T} - 1\right) = \left(\frac{\ell_B \lambda}{q} - 1\right) \equiv \left(\frac{\lambda}{\lambda_c} - 1\right).$$

For $\lambda < \lambda_c$, there is no solution of this equation.

For $\lambda \ge \lambda_c$, the solution is $\boxed{\frac{n_0 q^2 a^2}{2\varepsilon k_B T} = \left(\frac{\lambda}{\lambda_c} - 1\right)^2}$.

(g) For the λ above the critical density calculate the overall density of bound screening charge,

$$Q_{screening} = \int_a^\infty (dr)\rho(r).$$

(g) (2 points) We can proceed with the solution for the screening charge density for the case $\lambda > \lambda_c$:

$$\rho(r) = q n_0 e^{-\frac{q\Phi}{k_B T}} = q n_0 \left(\frac{a}{r}\right)^2 \frac{1}{\left(1 + \sqrt{\frac{n_0 q^2 a^2}{2\varepsilon k_B T}}\ln\frac{r}{a}\right)^2},$$ so the screening charge is,

$$Q_{screening} = 2\pi\int_a^\infty dr\, r\rho(r) = 2\pi q n_0 a^2 \int_0^\infty du \frac{1}{\left(1 + \sqrt{\frac{n_0 q^2 a^2}{2\varepsilon k_B T}}u\right)^2} = 2\pi a\sqrt{2\varepsilon k_B T n_0} = \boxed{(\lambda - \lambda_c)},$$

a result which was anticipated in the long comment at part (e).

Note: You will find that $Q_{screening} = \lambda - \frac{q}{\ell_B}$, which has the interpretation that,
above the critical line density of charge, there is sufficient screening so that, looked at from a very long distance, the line charge and its screening cloud always has an unscreened charge per unit length of q/ℓ_B. This is sometimes referred to as "charge renormalisation." You will recall that, in 1D, all the charge was screened. In 3D, all the charge escapes to infinity. This peculiar behavior is specific to 2D and is called Manning condensation.
For line charges below the critical value, all the counterions escape to infinity and there is not screening. I discussed this in Tutorial 12.
To see this in more detail, it is possible to carry through this solution in a geometry in which the whole system is put into a cylindrical "box" defined by a hard wall at r=R. As the box boundary $R \to \infty$, we get back to the case studied here. The advantage of the box is that the counterions do not get lost (they always remain inside the box), but you can see for the case $\lambda < \lambda_c$ that they recede off to large distances as R gets large, so there is no counterion cloud remaining in the vicinity of the original charge.